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SU(1,1) intelligent states: analytic representation in the unit disk

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Abstract. Intelligent states of the SU(1,1) Lie group are investigated using the analytic representation in the unit disk of the SU(1,1) coherent-state basis. By developing this representation, we study a special class of states, which are both intelligent and coherent. These states can be created using Hamiltonians, for which SU(1,1) is the dynamical symmetry group.

1. Introduction

The conceptions of coherent states (CS) and intelligent states (IS) have been developed from the Glauber CS $|\alpha\rangle$ [1]. These states form an overcomplete set and are eigenstates of the boson annihilation operator $\hat{a}$. Hence, the Heisenberg uncertainty relation for the harmonic oscillator position and momentum observables $\hat{x}$ and $\hat{p}$ is an equality over the Glauber states $|\alpha\rangle$. On the other hand, these states can be constructed by the action of elements of the Heisenberg–Weyl Lie group [2] on the vacuum state. This property has been used by Perelomov [3] and Gilmore [4], who have generalized the conception of CS for an arbitrary Lie group. Since generalized CS are obtained by the action of group elements on an extreme state of the group Hilbert space, they can be created using Hamiltonians, for which a given Lie group is the dynamical symmetry group.

Another property of the Glauber CS $|\alpha\rangle$ is the equality in the Heisenberg uncertainty relation, which can be generalized in order to define IS. For any two observables $\hat{A}$ and $\hat{B}$, the uncertainty relation is given by

$$(\Delta A)^2(\Delta B)^2 \geqslant \tfrac{1}{4}|\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle|^2 \tag{1.1}$$

where the variance $(\Delta A)^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$, $(\Delta B)^2$ is defined similarly, and the expectation values are taken over the state of the system $|\Psi\rangle$. States for which an equality is achieved in (1.1) are called IS [5]. States which also minimize the uncertainty product in (1.1) are called minimum-uncertainty states (MUS). For position and momentum observables $\hat{x}$ and $\hat{p}$, the commutation relation gives a constant: $[\hat{x},\hat{p}] = \mathrm{i}\hbar$, so, the right-hand side of (1.1) is state-independent and the IS and MUS, in this case, coincide and are given by the Glauber CS $|\alpha\rangle$. However, in the general case, $[\hat{A},\hat{B}] = \mathrm{i}\hat{C}$, where $\hat{C}$ is an operator, and then the IS and MUS are generally different. Intelligent states $|\lambda\rangle_{AB}$ for operators $\hat{A}$ and $\hat{B}$ are determined from the eigenvalue equation [6]

$$(\hat{A} + \mathrm{i}\gamma\hat{B})|\lambda\rangle_{AB} = \lambda|\lambda\rangle_{AB} \tag{1.2}$$

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where λ is a complex eigenvalue

$$\lambda = \langle\hat{A}\rangle + \mathrm{i}\gamma\langle\hat{B}\rangle \tag{1.3}$$

and γ is a real parameter given by

$$|\gamma| = \frac{\Delta A}{\Delta B} = \frac{|\langle\hat{C}\rangle|}{2(\Delta B)^2} = \frac{2(\Delta A)^2}{|\langle\hat{C}\rangle|}. \tag{1.4}$$

There is great interest in IS, especially for SU(2) and SU(1,1) Lie groups. Intelligent states were first defined in the present way by Aragone *et al* [5] for SU(2) Hermitian generators. These SU(2) IS have been shown recently to be useful for improving interferometric measurements [7]. A special case of SU(1,1) IS (for amplitude-squared boson operators) has recently been discussed [8]. The conception of squeezing is naturally related to SU(2) and SU(1,1) IS [9, 10]. We see from equation (1.4) that, for $|\gamma| > 1$, IS are squeezed in $\hat{B}$ and, for $|\gamma| < 1$, IS are squeezed in $\hat{A}$. Hence, IS can be useful for reducing quantum fluctuations [9]. In particular, SU(2) and SU(1,1) IS can be useful in quantum optics for improving the precision of measurements [7].

Unlike the special case of the Heisenberg–Weyl Lie group, IS and CS are generally different for arbitrary Lie groups, e.g. for the SU(2) and SU(1,1). However, even in these cases, there are some CS which are simultaneously IS, i.e. an intersection occurs between these two types of states. This intersection is of special importance in physics because IS, which are also coherent, can be created using Hamiltonians, for which a given Lie group is the dynamical symmetry group. It is not so for arbitrary IS, since, in general, SU(2) and SU(1,1) IS are constructed by using non-unitary operators [5, 9]. The relations between SU(2) IS and CS have been obtained in a complicated way by Aragone *et al* [5]. The aim of the present work is to develop a simple and effective method for relating SU(1,1) IS and CS. This general formalism is derived by using the standard representation of the SU(1,1) in the Hilbert space of entire functions, which are analytic in the unit disk. Similar procedures can be applied not only to the SU(1,1), but also to other Lie groups.

2. Properties of the SU(1,1) group and CS

In this section, we briefly discuss general properties of the SU(1,1) Lie group and the corresponding CS. The group SU(1,1) is the most elementary non-compact non-Abelian Lie group. It has several series of unitary irreducible representations: discrete, continuous and supplementary [11]. In the present work, we discuss only the case of the discrete series, which has many well known physical applications [3]. The Lie algebra corresponding to the group SU(1,1) has three generators: $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_3$, with the following commutation relations between them:

$$[\hat{K}_3, \hat{K}_\pm] = \pm\hat{K}_\pm \qquad [\hat{K}_-, \hat{K}_+] = 2\hat{K}_3. \tag{2.1}$$

Another basis of generators is appropriate:

$$\hat{K}_3 \qquad \hat{K}_1 = \frac{1}{2\mathrm{i}}(\hat{K}_- - \hat{K}_+) \qquad \hat{K}_2 = \frac{1}{2}(\hat{K}_- + \hat{K}_+) \tag{2.2}$$

where slightly different definitions of $\hat{K}_1$ and $\hat{K}_2$ are also possible according to different conventions. The commutation relations are the same for all conventional definitions of $\hat{K}_1$ and $\hat{K}_2$:

$$[\hat{K}_1, \hat{K}_2] = -\mathrm{i}\hat{K}_3 \qquad [\hat{K}_2, \hat{K}_3] = \mathrm{i}\hat{K}_1 \qquad [\hat{K}_3, \hat{K}_1] = \mathrm{i}\hat{K}_2. \tag{2.3}$$

The Casimir operator

$$\hat{Q} = \hat{K}_3^2 - \hat{K}_1^2 - \hat{K}_2^2 = \hat{K}_3^2 - \tfrac{1}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+) \tag{2.4}$$

for any irreducible representation is the identity operator multipied by a number

$$\hat{Q} = k(k-1)\hat{1}. \tag{2.5}$$

Thus, a representation of the SU(1,1) is determined by a single number k; for the discrete series, this number acquires discrete values $k = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. (The corresponding representations of the so-called universal covering group $\widetilde{\mathrm{SU}}(1,1)$ are also given by a single number k, but there it goes continuously from 0 to ∞.) The corresponding state space is spanned by the complete orthonormal basis $|k, m\rangle$ ($m = 0, 1, \ldots, \infty$)

$$\langle k, m|k, m'\rangle = \delta_{mm'} \qquad \sum_{m=0}^{\infty} |k, m\rangle\langle k, m| = \hat{1}. \tag{2.6}$$

These states may be defined by the following relations:

$$\begin{aligned} \hat{K}_3|k, m\rangle &= (k+m)|k, m\rangle \\ \hat{K}_+|k, m\rangle &= [(m+1)(2k+m)]^{1/2}|k, m+1\rangle \\ \hat{K}_-|k, m\rangle &= [m(2k+m-1)]^{1/2}|k, m-1\rangle. \end{aligned} \tag{2.7}$$

The SU(1,1) discrete series CS [3] are specified by pseudo-Euclidian unit vectors of the form

$$\boldsymbol{n} = (\cosh\tau, \sinh\tau\cos\varphi, \sinh\tau\sin\varphi). \tag{2.8}$$

The CS $|k, \zeta\rangle$ are obtained by applying unitary operators $\hat{D}(\xi)$ to the extreme state of the orthonormal basis $|k, m = 0\rangle$:

$$|k, \zeta\rangle = \exp(\xi\hat{K}_+ - \xi^*\hat{K}_-)|k, 0\rangle = (1 - |\zeta|^2)^k \exp(\zeta\hat{K}_+)|k, 0\rangle \tag{2.9}$$

where $\xi = -\frac{\tau}{2}\mathrm{e}^{-\mathrm{i}\varphi}$ and $\zeta = -\tanh\frac{\tau}{2}\mathrm{e}^{-\mathrm{i}\varphi}$, so $|\zeta| < 1$. Expanding the exponential and using (2.7), one obtains the decomposition of the CS over the orthonormal basis:

$$|k, \zeta\rangle = (1 - |\zeta|^2)^k \sum_{m=0}^{\infty} \left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right]^{1/2} \zeta^m |k, m\rangle. \tag{2.10}$$

The dynamics of the SU(1,1) CS has been studied by Gerry [12] who have derived the most general Hamiltonian which preserves the SU(1,1) CS under time evolution. The evolution of quantum systems driven by Hamiltonians which are linear combinations of the SU(1,1) generators has been considered in an excellent review [13], by using algebraic operatorial ordering methods.

The condition $|\zeta| < 1$ means that the 'phase space' of the SU(1,1) CS is the interior of the unit disk. The CS are normalized but not orthogonal to each other:

$$\langle k, \zeta_1 | k, \zeta_2\rangle = (1 - |\zeta_1|^2)^k (1 - |\zeta_2|^2)^k (1 - \zeta_1^* \zeta_2)^{-2k}. \tag{2.11}$$

The identity resolution is an important property of the CS:

$$\int \mathrm{d}\mu(\zeta)\, |k, \zeta\rangle\langle k, \zeta| = \hat{1} \tag{2.12}$$

where

$$\mathrm{d}\mu(\zeta) = \frac{2k-1}{\pi} \frac{\mathrm{d}^2\zeta}{(1-|\zeta|^2)^2} \tag{2.13}$$

and, for $k = \frac{1}{2}$, the limit $k \to \frac{1}{2}$ must be taken after the integration is carried out in the general form. Thus, the SU(1,1) CS form an overcomplete basis. One can introduce a non-normalized version of the CS

$$|k, \zeta\rangle\rangle = (1 - |\zeta|^2)^{-k} |k, \zeta\rangle = \sum_{m=0}^{\infty} \left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)} \right]^{1/2} \zeta^m |k, m\rangle. \tag{2.14}$$

Then one can construct the Hilbert space of entire functions $f(k, \zeta)$, which are analytic in the unit disk [11]. For a state

$$|f\rangle = \sum_{m=0}^{\infty} C_m(f) |k, m\rangle \tag{2.15}$$

we obtain

$$f(k, \zeta) = \langle\langle k, \zeta^* | f\rangle = \sum_{m=0}^{\infty} C_m(f) \left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)} \right]^{1/2} \zeta^m \tag{2.16}$$

and this state can be represented in the CS basis:

$$|f\rangle = \int \mathrm{d}\mu(\zeta)\, (1 - |\zeta|^2)^k f(k, \zeta^*) |k, \zeta\rangle. \tag{2.17}$$

We will refer to such a representation as the representation in the unit disk. The scalar product of two states is

$$\langle g|f\rangle = \int \mathrm{d}\mu(\zeta)\, (1 - |\zeta|^2)^{2k} f(k, \zeta^*) [g(k, \zeta^*)]^* \tag{2.18}$$

and the normalization condition for the state $|f\rangle$ is

$$\int \mathrm{d}\mu(\zeta)\, (1 - |\zeta|^2)^{2k} |f(k, \zeta^*)|^2 = 1. \tag{2.19}$$

The orthonormal basis in the Hilbert space of entire functions is given by

$$u_m(k,\zeta) = \langle\langle k,\zeta^*|k,m\rangle = \left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right]^{1/2}\zeta^m \tag{2.20}$$

and then we obtain the following useful formula:

$$\int \mathrm{d}\mu(\zeta)\,(1-|\zeta|^2)^{2k}\zeta^m\zeta^{*n} = \frac{m!\Gamma(2k)}{\Gamma(m+2k)}\delta_{mn}. \tag{2.21}$$

A CS $|k,\zeta_0\rangle$ is represented by the function

$$F(k,\zeta_0,\zeta) = \langle\langle k,\zeta^*|k,\zeta_0\rangle = (1-|\zeta_0|^2)^k(1-\zeta\zeta_0)^{-2k}. \tag{2.22}$$

The generators $\hat{K}_\pm$ and $\hat{K}_3$ act on the Hilbert space of entire functions as first-order differential operators:

$$\hat{K}_+ = \zeta^2\frac{\mathrm{d}}{\mathrm{d}\zeta} + 2k\zeta \qquad \hat{K}_- = \frac{\mathrm{d}}{\mathrm{d}\zeta} \qquad \hat{K}_3 = \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} + k. \tag{2.23}$$

By using expression (2.10), the following expectation values can be calculated over the SU(1,1) CS [9]:

$$\begin{aligned}
(\Delta\hat{K}_1)^2 &= \frac{k}{2(1-|\zeta|^2)^2}(1+|\zeta|^4-\zeta^{*2}-\zeta^2) = \frac{k}{2}(1+\sinh^2\tau\sin^2\varphi)\\
(\Delta\hat{K}_2)^2 &= \frac{k}{2(1-|\zeta|^2)^2}(1+|\zeta|^4+\zeta^{*2}+\zeta^2) = \frac{k}{2}(1+\sinh^2\tau\cos^2\varphi)\\
|\langle\hat{K}_3\rangle| &= k\frac{1+|\zeta|^2}{1-|\zeta|^2} = k\cosh\tau.
\end{aligned} \tag{2.24}$$

Wodkiewicz and Eberly [9] have pointed out that an equality holds in the uncertainty relation

$$(\Delta\hat{K}_1)^2(\Delta\hat{K}_2)^2 \geqslant \tfrac{1}{4}|\langle\hat{K}_3\rangle|^2 \tag{2.25}$$

when $\varphi = 0, \pi/2, \pi, 3\pi/2, \ldots$. If $\varphi = 0, \pi, 2\pi, \ldots$, then ζ is real and $|k,\zeta\rangle$ is squeezed in $\hat{K}_1$. If $\varphi = \pi/2, 3\pi/2, 5\pi/2, \ldots$, then ζ is pure imaginary and $|k,\zeta\rangle$ is squeezed in $\hat{K}_2$. (We note that in our convention (2.2) $\hat{K}_1 = (\hat{K}_- - \hat{K}_+)/2\mathrm{i}$, $\hat{K}_2 = (\hat{K}_- + \hat{K}_+)/2$, while, in the convention of Wodkiewicz and Eberly [9], $\hat{K}_1 = (\hat{K}_+ + \hat{K}_-)/2$, $\hat{K}_2 = (\hat{K}_+ - \hat{K}_-)/2\mathrm{i}$.) We want to find the analytic representations of SU(1,1) IS in the unit disk, in order to obtain a convenient way of calculating their various properties. In particular, we will be able to determine when an intelligent state is also coherent.

3. The analytic representation of IS

By using the Hilbert space of entire functions discussed in the preceding section, we can write eigenvalue equations of type (1.2) for the SU(1,1) Hermitian generators $\hat{K}_i$ ($i = 1,2,3$) as first-order differential equations. Solutions of these equations are entire analytic functions, which represent SU(1,1) IS. Equation (1.2) becomes

$$(\hat{K}_i + \mathrm{i}\gamma\hat{K}_j)|\lambda\rangle_{ij} = \lambda|\lambda\rangle_{ij} \qquad i,j = 1,2,3 \qquad (i \neq j) \tag{3.1}$$

so $|\lambda\rangle_{ij}$ are the IS for operators $\hat{K}_i$ and $\hat{K}_j$ (the SU(1, 1) IS). Defining

$$\Lambda_{ij}(k,\lambda,\gamma,\zeta) = \langle\langle k,\zeta^*|\lambda\rangle_{ij} \tag{3.2}$$

we obtain

$$(\hat{K}_i + \mathrm{i}\gamma\hat{K}_j)\Lambda_{ij} = \lambda\Lambda_{ij} \tag{3.3}$$

where $\hat{K}_i$ and $\hat{K}_j$ are first-order differential operators obtained from (2.23). We start by considering the $\hat{K}_2$–$\hat{K}_1$ IS. We write

$$\hat{K}_2 + \mathrm{i}\gamma\hat{K}_1 = \frac{1-\gamma}{2}\hat{K}_+ + \frac{1+\gamma}{2}\hat{K}_- = \beta_-\hat{K}_+ + \beta_+\hat{K}_- \tag{3.4}$$

where $\beta_\pm = (1\pm\gamma)/2$. Then equation (3.3) is

$$(\beta_-\zeta^2 + \beta_+)\frac{\mathrm{d}\Lambda_{21}}{\mathrm{d}\zeta} = (\lambda - 2k\beta_-\zeta)\Lambda_{21}. \tag{3.5}$$

When integrating this equation, we must treat the cases of $\beta_- = 0$ ($\gamma = 1$) and $\beta_+ = 0$ ($\gamma = -1$) separately.

For $\beta_- = 0$ ($\gamma = 1$) we obtain eigenstates of the operator $\hat{K}_-$. We define

$$Z(k,z,\zeta) \equiv \Lambda_{21}(k,\lambda = z,\gamma = 1,\zeta) \tag{3.6}$$

and then equation (3.5) is

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}Z(k,z,\zeta) = zZ(k,z,\zeta) \tag{3.7}$$

with the simple solution

$$Z(k,z,\zeta) = \mathcal{N}\exp(z\zeta). \tag{3.8}$$

Here, and in the following, $\mathcal{N}$ denotes a normalization factor. By using the normalization condition (2.19), we get, up to an unimportant phase factor,

$$\mathcal{N}^{-2} = \int \mathrm{d}\mu(\zeta)\,(1-|\zeta|^2)^{2k}\mathrm{e}^{z\zeta^*}\mathrm{e}^{z^*\zeta}. \tag{3.9}$$

Expanding the exponentials and using orthogonality relation (2.21), we obtain

$$\mathcal{N} = \frac{|z|^{k-1/2}}{[I_{2k-1}(2|z|)\Gamma(2k)]^{1/2}} \tag{3.10}$$

where I_ν is the ν-order modified Bessel function of the first kind. The eigenstates of the lowering operator $\hat{K}_-$ were first constructed by Barut and Girardello [14], who wrote them in the orthonormal basis

$$|k,z\rangle = |\lambda(\gamma = 1)\rangle_{21} = \sum_{m=0}^{\infty} C_m(k,z)|k,m\rangle. \tag{3.11}$$

The coefficients $C_m(k, z)$ can be calculated using the power-series expansion of $Z(k, z, \zeta)$ in ζ and comparing this expansion with the general form (2.16). The result is

$$C_m(k, z) = \frac{|z|^{k-1/2} z^m}{[I_{2k-1}(2|z|) m! \Gamma(2k+m)]^{1/2}}. \tag{3.12}$$

As has been shown by Barut and Girardello [14], the states $|k, z\rangle$ form an overcomplete basis with the identity resolution

$$\frac{2}{\pi} \int d^2 z \, I_{2k-1}(2|z|) K_{2k-1}(2|z|) |k, z\rangle\langle k, z| = \hat{1} \tag{3.13}$$

where K_ν is the ν-order modified Bessel function of the second kind. Thus, these states can be used to represent a state $|f\rangle$ belonging to the SU(1,1) discrete series state space, and the corresponding representation can be constructed in the Hilbert space of entire functions, which are analytic over the whole z-plane. This analytic representation, based on Barut–Girardello states, can be derived for any SU(1,1) IS [15]. The Barut–Girardello states $|k, z\rangle$ and the SU(1,1) CS $|k, \zeta\rangle$ never coincide, excluding the trivial case $z = 0$ and $\zeta = 0$ when both types of states degenerate to the extreme state of the orthonormal basis $|k, m = 0\rangle$.

Now we take the case $\beta_+ = 0$ ($\gamma = -1$). Then equation (3.5) is

$$\zeta^2 \frac{d}{d\zeta} \Lambda_{21}(k, \lambda, \gamma = -1, \zeta) = [\lambda - 2k\zeta] \Lambda_{21}(k, \lambda, \gamma = -1, \zeta) \tag{3.14}$$

with the solution

$$\Lambda_{21}(k, \lambda, \gamma = -1, \zeta) = \mathcal{N} \zeta^{-2k} \exp(-\lambda/\zeta). \tag{3.15}$$

We see that this function is not analytic, that is, for $\gamma = -1$ we cannot obtain a proper solution. From normalization condition (2.19) we obtain

$$\mathcal{N}^{-2} = \int d\mu(\zeta) \, (1 - |\zeta|^2)^{2k} |\zeta|^{-4k} \exp\left(-\frac{\lambda}{\zeta^*} - \frac{\lambda^*}{\zeta}\right). \tag{3.16}$$

It is not difficult to see that the integral over $|\zeta|$ diverges for all possible values of k and λ. Thus, there are no eigenstates of the raising operator $\hat{K}_+$.

Now we integrate equation (3.5) for arbitrary real γ, excluding $\gamma = \pm 1$. The solution is

$$\Lambda_{21}(k, \lambda, \gamma, \zeta) = \mathcal{N} \left[\sqrt{\frac{\gamma+1}{\gamma-1}} + \zeta\right]^{-k+\lambda/\sqrt{\gamma^2-1}} \left[\sqrt{\frac{\gamma+1}{\gamma-1}} - \zeta\right]^{-k-\lambda/\sqrt{\gamma^2-1}}. \tag{3.17}$$

The condition of the analyticity requires

$$\left|\sqrt{\frac{\gamma+1}{\gamma-1}}\right| \geqslant 1 \qquad \text{i.e. } \gamma \geqslant 0. \tag{3.18}$$

The eigenstate of the operator $\hat{K}_2$ with eigenvalue λ is represented by the function

$$\Lambda_{21}(k, \lambda, \gamma = 0, \zeta) = \mathcal{N} (i + \zeta)^{-k-i\lambda} (i - \zeta)^{-k+i\lambda}. \tag{3.19}$$

By taking $\lambda = \mathrm{i}\gamma\tilde{\lambda}$, we obtain, in the limit $\gamma \to \infty$, the function that represents the eigenstate of the operator $\hat{K}_1$ with eigenvalue $\tilde{\lambda}$:

$$\Lambda_{21}(k, \lambda, \gamma \to \infty, \zeta) \to (1+\zeta)^{-k+\mathrm{i}\tilde{\lambda}}(1-\zeta)^{-k-\mathrm{i}\tilde{\lambda}}. \tag{3.20}$$

We can compare the function Λ_{21} $(k, \lambda, \gamma, \zeta)$ of equation (3.17) with the function $F(k, \zeta_0, \zeta)$, which represents a CS $|k, \zeta_0\rangle$ (see equation (2.22)). We see that a $\hat{K}_2$–$\hat{K}_1$ intelligent state is also coherent when

$$\lambda = \pm k\sqrt{\gamma^2 - 1}. \tag{3.21}$$

Then

$$\zeta_0 = \pm\sqrt{\frac{\gamma - 1}{\gamma + 1}} \tag{3.22}$$

respectively, and the condition $|\zeta_0| < 1$ requires $0 < \gamma < \infty$, in accordance with analyticity condition (3.18). When $\gamma > 1$ (squeezing in $\hat{K}_1$), λ and ζ_0 are real. When $\gamma < 1$ (squeezing in $\hat{K}_2$), λ and ζ_0 are pure imaginary. When the SU(1,1) CS and $\hat{K}_2$–$\hat{K}_1$ IS coincide, the normalization factor $\mathcal{N}$ in equation (3.18) is identified, up to an unimportant phase factor, to be

$$\mathcal{N} = \left(\frac{1 - |\zeta_0|^2}{|\zeta_0|^2}\right)^k. \tag{3.23}$$

We see that the comparison of states can be conveniently achieved by the comparison of corresponding analytic functions, which represent states in the unit disk.

We continue by considering the $\hat{K}_1$–$\hat{K}_3$ IS. The following equation:

$$[\hat{K}_1 + \mathrm{i}\gamma\hat{K}_3]\Lambda_{13}(k, \lambda, \gamma, \zeta) = \lambda\Lambda_{13}(k, \lambda, \gamma, \zeta) \tag{3.24}$$

can be rewritten in the form

$$(\zeta^2 - 1 + 2\gamma\zeta)\frac{\mathrm{d}\Lambda_{13}}{\mathrm{d}\zeta} = -2(\mathrm{i}\lambda + k\gamma + k\zeta)\Lambda_{13}. \tag{3.25}$$

By integrating this equation, we find

$$\Lambda_{13}(k, \lambda, \gamma, \zeta) = \mathcal{N}\left[\zeta + \gamma - \sqrt{\gamma^2+1}\right]^{-k-\mathrm{i}\lambda/\sqrt{\gamma^2+1}}\left[\zeta + \gamma + \sqrt{\gamma^2+1}\right]^{-k+\mathrm{i}\lambda/\sqrt{\gamma^2+1}}. \tag{3.26}$$

The analyticity condition in this case is somewhat complicated:

$$\begin{aligned} &\left|\gamma + \sqrt{\gamma^2+1}\right| \geqslant 1 \qquad \text{i.e. } \gamma \geqslant 0 \text{ for } \operatorname{Im}\lambda/\sqrt{\gamma^2+1} \geqslant k \\ &\left|\gamma - \sqrt{\gamma^2+1}\right| \geqslant 1 \qquad \text{i.e. } \gamma \leqslant 0 \text{ for } \operatorname{Im}\lambda/\sqrt{\gamma^2+1} \leqslant -k. \end{aligned} \tag{3.27}$$

For $-k < \mathrm{Im}\,\lambda/\sqrt{\gamma^2+1} < k$, only $\gamma = 0$ guarantees the analyticity of Λ_{13}. By comparing the function $\Lambda_{13}(k,\lambda,\gamma,\zeta)$ of equation (3.26) with the function $F(k,\zeta_0,\zeta)$ of equation (2.22), we see that a $\hat{K}_1$–$\hat{K}_3$ IS is also coherent when

$$\lambda = \pm \mathrm{i}k\sqrt{\gamma^2+1}. \tag{3.28}$$

Then

$$\zeta_0 = \frac{-1}{\gamma \pm \sqrt{\gamma^2+1}} \tag{3.29}$$

respectively, and the condition $|\zeta_0| < 1$ requires $0 < \gamma < \infty$ for the upper sign and $0 > \gamma > -\infty$ for the lower sign, in accordance with analyticity condition (3.27). For the upper sign, λ lies on the upper imaginary half-axis and ζ_0 is real and negative. For the lower sign, λ lies on the lower imaginary half-axis and ζ_0 is real and positive. When $|\zeta_0| > 0.414$, then $|\gamma| < 1$ and the squeezing is obtained in $\hat{K}_1$. When $|\zeta_0| < 0.414$, then $|\gamma| > 1$ and the squeezing is obtained in $\hat{K}_3$. When the SU(1,1) CS and $\hat{K}_1$–$\hat{K}_3$ IS coincide, the normalization factor $\mathcal{N}$ in equation (3.26) is given, up to an unimportant phase factor, by expression (3.23), but with ζ_0 of equation (3.29).

Now we consider the case of the $\hat{K}_2$-$\hat{K}_3$ IS. The following equation:

$$[\hat{K}_2 + \mathrm{i}\gamma\hat{K}_3]\Lambda_{23}(k,\lambda,\gamma,\zeta) = \lambda\Lambda_{23}(k,\lambda,\gamma,\zeta) \tag{3.30}$$

can be rewritten in the form

$$(\zeta^2 + 1 + 2\mathrm{i}\gamma\zeta)\frac{\mathrm{d}\Lambda_{23}}{\mathrm{d}\zeta} = 2(\lambda - \mathrm{i}k\gamma - k\zeta)\Lambda_{23}. \tag{3.31}$$

The solution is

$$\Lambda_{23}(k,\lambda,\gamma,\zeta) = \mathcal{N}\left[\zeta + \mathrm{i}\gamma - \mathrm{i}\sqrt{\gamma^2+1}\right]^{-k-\mathrm{i}\lambda/\sqrt{\gamma^2+1}}\left[\zeta + \mathrm{i}\gamma + \mathrm{i}\sqrt{\gamma^2+1}\right]^{-k+\mathrm{i}\lambda/\sqrt{\gamma^2+1}} \tag{3.32}$$

and the analyticity condition in this case is the same as before (3.27). In the usual way, we compare the function $\Lambda_{23}(k,\lambda,\gamma,\zeta)$ with the function $F(k,\zeta_0,\zeta)$ of equation (2.22) in order to find when a $\hat{K}_2$–$\hat{K}_3$ IS is also coherent. We obtain the condition

$$\lambda = \pm \mathrm{i}k\sqrt{\gamma^2+1} \tag{3.33}$$

exactly as in the previous case. Then

$$\zeta_0 = \frac{\mathrm{i}}{\gamma \pm \sqrt{\gamma^2+1}} \tag{3.34}$$

respectively, and the condition $|\zeta_0| < 1$ requires $0 < \gamma < \infty$ for the upper sign and $0 > \gamma > -\infty$ for the lower sign, in accordance with analyticity condition (3.27). For the upper sign, λ and ζ_0 lie on the upper imaginary half-axis while, for the lower sign, λ and ζ_0 lie on the lower imaginary half-axis. When $|\zeta_0| > 0.414$, then $|\gamma| < 1$ and the squeezing is obtained in $\hat{K}_2$. When $|\zeta_0| < 0.414$, then $|\gamma| > 1$ and the squeezing is obtained in $\hat{K}_3$. The normalization factor $\mathcal{N}$ is the same as before.

If one knows entire functions which represent given states, then various eigenvalues of different operators can be calculated over these states. Since $\Lambda_{ij}(k,\lambda,\gamma,\zeta)$ are entire analytic functions of ζ, they can be expanded into the power series

$$\Lambda_{ij}(k,\lambda,\gamma,\zeta)=\sum_{m=0}^{\infty}L_{ij}^{(m)}(k,\lambda,\gamma)\zeta^{m}. \tag{3.35}$$

By using the decomposition of the SU(1,1) IS over the orthonormal basis

$$|\lambda\rangle_{ij}=\sum_{m=0}^{\infty}C_{ij}^{(m)}(k,\lambda,\gamma)|k,m\rangle \tag{3.36}$$

we can write, similarly to general form (2.16),

$$\Lambda_{ij}(k,\lambda,\gamma,\zeta)=\sum_{m=0}^{\infty}C_{ij}^{(m)}(k,\lambda,\gamma)\left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right]^{1/2}\zeta^{m}. \tag{3.37}$$

This expansion gives the relation between the coefficients $L_{ij}^{(m)}(k,\lambda,\gamma)$ and $C_{ij}^{(m)}(k,\lambda,\gamma)$. Then expectation values of an operator $\hat{A}$ over the SU(1,1) IS are given by

$$\begin{aligned}{}_{ij}\langle\lambda|\hat{A}|\lambda\rangle_{ij}=\sum_{n,m=0}^{\infty}\langle k,n|\hat{A}|k,m\rangle\left[\frac{n!m!}{\Gamma(n+2k)\Gamma(m+2k)}\right]^{1/2}\\ \times\Gamma(2k)\left[L_{ij}^{(n)}(k,\lambda,\gamma)\right]^{*}L_{ij}^{(m)}(k,\lambda,\gamma).\end{aligned} \tag{3.38}$$

Thus, we can calculate expectation values of different operators over the SU(1,1) IS if we know the corresponding analytic functions and their power-series expansions.

4. Discussion and conclusions

We have established the analytic representation for all types of the SU(1,1) IS in the CS basis. In this representation, the SU(1,1) IS are associated with the entire functions, which are analytic in the unit ζ disk. By investigating these functions, we have found the important class of the SU(1,1) IS; an intelligent state from this class is simultaneously coherent, so such a state can be created by using Hamiltonians for which SU(1,1) is the group of dynamical symmetry. Such Hamiltonians are well known in quantum optics [12, 13] and some of them are related to nonlinear optical devices such as parametric amplifiers and four-wave mixers. When these nonlinear devices are used in interferometry, measurements can be improved by applying an appropriate intelligent state of the light field. In order to be able to create these states of light, one must choose states which lie in the intersection of the SU(1,1) IS and CS. So, optical schemes can be constructed where a reduction of quantum fluctuations is possible. Besides these quantum optical applications, the investigation of the analytic representation of the SU(1,1) IS in the unit disk of the CS basis is interesting from the purely theoretical point of view. This representation of the SU(1,1) IS provides us with a convenient way of calculating various properties of the IS in different physical realizations of the SU(1,1) Lie group. It is important to note that, since an analytic representation in the generalized CS basis can be developed for an arbitrary Lie group, properties of corresponding IS can be studied, by using analytic representations, in a quite general way.

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References

[1] Glauber R J 1963 *Phys. Rev.* **130** 2529; **131** 2766
Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277
[2] Weyl H 1950 *The Theory of Groups and Quantum Mechanics* (New York: Dover)
[3] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222; 1977 *Sov. Phys.–Usp.* **20** 703; 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[4] Gilmore R 1972 *Ann. Phys.* **74** 391; 1974 *Rev. Mex. de Fisica* **23** 142; 1974 *J. Math. Phys.* **15** 2090
Zhang W-M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[5] Aragone C, Guerri G, Salamo S and Tani J L 1974 *J. Phys. A: Math. Nucl. Gen.* **7** L149
Aragone C, Chalbaud E and Salamo S 1976 *J. Math. Phys.* **17** 1963
see also Ruschin S and Ben-Aryeh Y 1976 *Phys. Lett.* **58A** 207
[6] Gottfried K 1966 *Quantum Mechanics* (New York: Benjamin)
[7] Hillery M and Mlodinow L 1993 *Phys. Rev.* A **48** 1548
[8] Yu D and Hillery M 1994 *Quantum Opt.* **6** 37
[9] Wodkiewicz K and Eberly J H 1985 *J. Opt. Soc. Am.* B **2** 458
[10] Hillery M 1987 *Phys. Rev.* A **36** 3796; 1989 *Phys. Rev.* A **40** 3147
[11] Bargmann V 1947 *Ann. Math.* **48** 568
Barut A O and Phillips C 1968 *Commun. Math. Phys.* **8** 52
Vilenkin N J 1968 *Special Functions and the Theory of Group Representations* (Providence, RI: American Mathematical Society)
[12] Gerry C C 1985 *Phys. Rev.* A **31** 2721; 1985 *Phys. Lett.* **109A** 149
[13] Dattoli G, Gallardo J C and Torre A 1988 *Riv. Nuovo Cimento* **11** and references therein
[14] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[15] Trifonov D A 1994 *J. Math. Phys.* **35** 2297